Problem set 7
Due date: 1st April

(Submit any five)

Exercise 59. Let $\{X_i\}_{i\in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$.

(1) Suppose $\sup_i \mathbf{E}[|X_i| h(|X_i|)]$ is finite, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function that goes to infinity. Show that $\{X_i\}_{i\in I}$ is uniformly integrable.
(2) If $\{X_i\}$ is u.i, show that $\{X_i\}$ is tight. Give a counterexample to the converse.

Exercise 60. (1) If $\{X_i\}_{i\in I}$ is u.i, then show that $\sup_i \mathbf{E}|X_i| < \infty$.
(2) If there is a non-negative random variable Y with $\mathbf{E}[Y] < \infty$ such that $|X_i| \le Y$ a.s. for all $i \in I$, then show that $\{X_i\}$ is u.i.

Exercise 61. Let X_n be i.i.d with $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = \frac{1}{2}$. Show that for any $\gamma > \frac{1}{2}$,

$$\frac{S_n}{n^\gamma} \xrightarrow{a.s.} 0.$$

[**Remark:** Try to imitate the proof of SLLN under fourth moment assumption. If you write the proof correctly, it should go for any random variable which has moments of all orders. You do not need to show this for the homework].

Exercise 62. Let X_n be independent real-valued random variables.

(1) Show by example that the event $\{\sum X_n$ converges to a number in [1,3]$\}$ can have probability strictly between 0 and 1.
(2) Show that the event $\{\sum X_n$ converges to a finite number$\}$ has probability zero or one.

Exercise 63. Let X_n be i.i.d exponential(1) random variables.

(1) If b_n is a sequence of numbers that converge to 0, show that $\limsup b_n X_n$ is a constant (a.s.). Find a sequence b_n so that $\limsup\, b_n X_n = 1$ a.s.
(2) Let M_n be the maximum of $X_1, \ldots, X_n$. If $a_n \to \infty$, show that $\limsup \frac{M_n}{a_n}$ is a constant (a.s.). Find a_n so that $\limsup \frac{M_n}{a_n} = 1$ (a.s.).

[**Remark:** Can you do the same if X_n are i.i.d N(0,1)? Need not show this for the homework, but note that the main ingredient is to find a simple expression for $\mathbf{P}(X_1 > t)$ asymptotically as $t \to \infty$].

Exercise 64. Let X_n be i.i.d real valued random variables with common distribution μ. For each n, define the random probabilty measure μ_n as $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{X_k}$. Let F_n be the CDF of μ_n. Show that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow{a.s.} 0 \quad a.s.$$